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Local Support Bases for a Class of Spline Functions

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Bases for a class of splines consisting piecewise of elements in the null space of a linear differential operator L with the component pieces tied smoothly together at the knots by requiring the continuity of certain Extended-Hermite–Birkhoff linear functionals are obtained. In particular, first, certain one-sided splines are constructed as linear combinations of an appropriate Green's function, and then local support splines are constructed as linear combinations of the one-sided splines. Finally, local support bases for a finite-dimensional space of splines are obtained.

1. INTRODUCTION

The purpose of this paper is to construct local support bases for spaces of spline functions where the splines consist piecewise of elements in the null space of a differential operator L with the component pieces tied smoothly together at the knots by requiring the continuity of certain Extended-Hermite–Birkhoff linear functionals. In order to proceed directly to a precise definition of the class of splines under consideration, we defer a discussion of previous work on local support splines to Section 5. We mention here only that such local bases are of use in collocation methods for the numerical

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solution of linear boundary-value problems [8], as well as in the construction of local approximation schemes [9, 13].

Let $\cdots < x_{-1} < x_0 < x_1 < \cdots$ be a bi-infinite set of distinct real numbers. Suppose $\mathcal{N} = \operatorname{span}\{u_i(x)\}_{i=1}^n$ is the null space (or set of fundamental solutions) of an nth-order linear differential operator L with leading coefficient 1. Suppose for each ν that $\Lambda_\nu = \{\lambda_{\nu i}\}_{i=1}^{l_\nu}$ is an Extended-Hermite–Birkhoff (EHB-) set of linear functionals with support at x_ν; i.e.,

$$\lambda_{\nu i} = \sum_{j=0}^{n-1} \gamma_{\nu ij} e_{x_\nu}^{(j)}, \qquad i = 1, 2,..., l_\nu,$$

where $e_{x_\nu}^{(j)}\varphi = \varphi^{(j)}(x_\nu)$. We assume that the $\lambda_{\nu 1},..., \lambda_{\nu l_\nu}$ are linearly independent in the sense that the matrices $\Gamma_\nu = (\gamma_{\nu ij})_{i=1,j=0}^{l_\nu, n-1}$ are of full rank l_ν. We are interested in determining local support bases for the class of splines

$$\begin{aligned} \mathcal{S} = \{s : s\,|_{I_\nu} = s_\nu\,|_{I_\nu} \qquad & \text{for some } s_\nu \in \mathcal{N}, \text{ all } \nu, \\ \lambda_{\nu i} s_{\nu-1} = \lambda_{\nu i} s_\nu, \qquad & i = 1, 2,..., l_\nu\}, \end{aligned} \tag{1.1}$$

where $I_\nu = (x_\nu, x_{\nu+1})$. The differentiations implicit in evaluating the $\lambda_{\nu i}$ on $s_{\nu-1}$ and s_ν are to be understood as the left and right derivatives at x_ν, respectively.

The class of splines $\mathcal{S}$ in (1.1) is quite general. It includes, for example, g-splines, L-splines, Lg-splines, and the generalized splines of Greville [7] and Jerome [9] (and thus the various polynomial-, trigonometric-, exponential-, and hyperbolic-splines, etc.). For other examples, see Section 5, and in particular, Remark 8 there.

We caution the reader not to confuse the sets Λ_ν of EHB-functionals describing smoothness of the splines in $\mathcal{S}$ with the sets of EHB-functionals which are often used to impose interpolation constraints in variational problems; they are in a sense dual sets. Specifically, for classes of splines arising from variational problems, the sets Λ_ν will include lower-order smoothness functionals as well as certain adjoint natural boundary functionals involving higher-order derivatives. We also note that in the definition of $\mathcal{S}$ we leave the value of the splines at the knots undefined (we could adopt the convention that $s(x_\nu) = s_\nu(x_\nu)$, for example).

2. Spline Bases with One-Sided Support

In this section we introduce some splines in $\mathcal{S}$ which will be the basic building blocks for local support splines. We need to introduce certain

Green's functions associated with $L = D^n + \sum_{j=0}^{n-1} a_j(x) D^j$. Suppose $a_j \in C^n(\mathbb{R})$, $j = 0, 1, \ldots, n-1$. Let L_i and $L_i{}^*$ be defined by

$$\begin{aligned} L_i\varphi &= \varphi^{(i)} + a_{n-1}\varphi^{(i-1)} + \cdots + a_{n-i}\varphi, \\ L_i{}^*\varphi &= (-1)^i \varphi^{(i)} + (-1)^{i-1} (a_{n-1}\varphi)^{(i-1)} + \cdots + a_{n-i}\varphi, \end{aligned} \tag{2.1}$$

for $i = 1, 2, \ldots, n$. For convenience set $L_0 = L_0{}^* = I$, the identity operator. $L_i{}^*$ is the formal adjoint of L_i, and $L_n = L$.

We denote by $\{v_i(x)\}_{i=1}^n$ the set of *adjunct* functions for L (cf. Greville [7]); they form a fundamental system for L^* (i.e., a basis for the null space of L^*) and are characterized by

$$\sum_{i=1}^{n} u_i^{(k)}(x)\, v_i(x) = \delta_{k,n-1}, \qquad k = 0, 1, \ldots, n-1. \tag{2.2}$$

This relation implies (cf. the Appendix)

$$\sum_{i=1}^{n} u_i^{(k)}(x)\, L_{n-j-1}^* v_i(x) = \delta_{k,j}, \qquad k = 0, 1, \ldots, n-1, \tag{2.3}$$

for $j = 0, 1, \ldots, n-1$. Notice that statement (2.3) is precisely the statement that $W^{-1}(u_1, \ldots, u_n)$ is the matrix $(W_{ij}^{-1}) = (L_{n-j}^* v_i)$, where the (i, j) entry of $W(u_1, \ldots, u_n)$ is $u_j^{(i)}$. Now for $j = 0, 1, \ldots, n-1$, define

$$\theta_j(x; \xi) = \sum_{i=1}^{n} u_i(x)\, L_{n-j-1}^* v_i(\xi) = L_{n-j-1}^* \theta_{n-1}(x; \xi), \tag{2.4}$$

where L_{n-j-1}^* operates on the ξ variable. Clearly, for $j = 0, 1, \ldots, n-1$,

$$D_x{}^k \theta_j(x; \xi)|_{x=\xi} = \delta_{k,j}, \qquad k = 0, 1, \ldots, n-1, \tag{2.5}$$

so that the functions

$$\hat{\theta}_j(x; \xi) = \begin{cases} \theta_j(x; \xi), & x \geqslant \xi, \\ 0, & x < \xi, \end{cases} \tag{2.6}$$

possess jumps of $\delta_{k,j}$ in their kth derivatives at ξ, for $j, k = 0, 1, \ldots, n-1$. These functions are thus similar to the Green's kernel for initial value problems, and in fact, $\hat{\theta}_{n-1}$ is the usual such Green's function. Moreover, $\hat{\theta}_\nu(x; \xi) = (x - \xi)_+^\nu/\nu!$, $\nu = 0, 1, \ldots, n-1$, for $L = D^n$.

Although the functions $\hat{\theta}_j(x; x_\nu)$ are zero for $x < x_\nu$ and are restrictions of functions in $\mathscr{N}$ for $x > x_\nu$, they are not necessarily in $\mathscr{S}$ because they may fail to satisfy the required ties at x_ν. However, by taking linear combinations of $\hat{\theta}_0(\cdot; x_\nu), \ldots, \hat{\theta}_{n-1}(\cdot; x_\nu)$, we can produce $m_\nu = n - l_\nu$ linearly independent spline functions in $\mathscr{S}$ with knot at x_ν. We have

LEMMA 2.1. *Suppose the matrices* $\Gamma_\nu = (\gamma_{\nu ij})_{i=1,j=0}^{l_\nu, n-1}$ *in the definition of* $\mathscr{S}$ *are of full rank* l_ν. *Let* $\alpha_{\nu 1},\ldots,\alpha_{\nu m_\nu}$ *be* $m_\nu = n - l_\nu$ *linearly independent vectors in* $\mathbb{R}^n$ *which satisfy*

$$\Gamma_\nu \alpha_{\nu k} = 0, \qquad k = 1, 2,\ldots, m_\nu. \tag{2.7}$$

Then, with $\alpha_{\nu k} = (\alpha_{\nu,k,0},\ldots,\alpha_{\nu,k,n-1})^T$, *the functions*

$$\rho_{\nu k}(x) = \sum_{\mu=0}^{n-1} \alpha_{\nu k \mu} \hat{\theta}_\mu(x; x_\nu), \qquad k = 1, 2,\ldots, m_\nu, \tag{2.8}$$

are m_ν *linearly independent splines in* $\mathscr{S}$. *Moreover, each of them is identically zero for* $x < x_\nu$.

Proof. Clearly, $\rho_{\nu k}$ has the correct piecewise structure; i.e., it is zero for $x < x_\nu$ and is an element of $\mathscr{N}$ for $x \geqslant x_\nu$. Since (2.7) forces the required continuities at the knot x_ν, $\rho_{\nu k} \in \mathscr{S}$. Now if

$$0 = \sum_{k=1}^{m_\nu} d_k \rho_{\nu k}(x) = \sum_{j=0}^{n-1} \hat{\theta}_j(x; x_\nu) \sum_{k=1}^{m_\nu} d_k \alpha_{\nu k j},$$

then a nontrivial linear combination of $\{\hat{\theta}_j\}_{j=0}^{n-1}$ would be zero if some $d_k \neq 0$. Thus, the constants d_k must be 0, and the $\{\rho_{\nu k}\}_{k=1}^{m_\nu}$ are linearly independent. ∎

By (2.4), (2.6), we may write (2.8) as

$$\rho_{\nu j}(x) = \begin{cases} 0, & x < x_\nu \\ [u_1(x),\ldots, u_n(x)] C_{\nu j}, & x \geqslant x_\nu, \end{cases} \tag{2.9}$$

where

$$C_{\nu j} = \begin{bmatrix} \sum_{\mu=0}^{n-1} \alpha_{\nu j \mu} L_{n-1-\mu}^* v_1(x_\nu) \\ \vdots \\ \sum_{\mu=0}^{n-1} \alpha_{\nu j \mu} L_{n-1-\mu}^* v_n(x_\nu) \end{bmatrix}. \tag{2.10}$$

3. LOCAL SUPPORT SPLINES

In this section we show how local support splines can be constructed as linear combinations of the one-sided splines described in Lemma 2.1 and Eqs. (2.9) and (2.10).

At times it will be convenient to have a single subscript ordering of the ρ's. We define

$$\ldots, \rho_{-1}, \rho_0, \rho_1, \ldots$$

to be the lexicographical ordering of

$$\ldots, \rho_{0,1}, \ldots, \rho_{0,m_0}, \rho_{1,1}, \ldots, \rho_{1,m_1}, \rho_{2,1}, \ldots$$

with the identification $\rho_1 = \rho_{0,1}$. We use both index schemes in the sequel. We will also write

$$\ldots, C_{-1}, C_0, C_1, \ldots$$

for the corresponding lexicographical ordering of the vectors $C_{\nu j}$ corresponding to $\rho_{\nu j}$ in (2.9) and (2.10).

LEMMA 3.1. *Suppose for some* $i_1 < i_2 < \cdots < i_q$ *and some* $\delta^T = (\delta_1, \ldots, \delta_q) \in \mathbb{R}^q$ *that* $\tilde{C}\delta = 0$, *where* $\tilde{C}$ *is the* $n \times q$ *matrix* $\tilde{C} = (C_{i_1}, C_{i_2}, \ldots, C_{i_q})$. *Then*

$$B(x) = \sum_{\mu=1}^{q} \delta_\mu \rho_{i_\mu}(x) \tag{3.1}$$

is a spline in $\mathscr{S}$. *Moreover, if* ρ_{i_1} *is associated with the knot* x_l *and* ρ_{i_q} *is associated with the knot* x_r, *then* $B(x)$ *is identically zero outside of* $[x_l, x_r]$.

Proof. Clearly, $B \in \mathscr{S}$ since $\{\rho_{i_\nu}\}_\nu \subset \mathscr{S}$. Moreover, $\rho_{i_1}(x) = 0$ for $x < x_l$, and by the ordering, the same is true for $\rho_{i_2}, \ldots, \rho_{i_q}$. Thus $B(x) \equiv 0$ for $x < x_l$. Now for $x \geqslant x_r$, each of the functions ρ_{i_μ} is given by (2.9); i.e.,

$$\rho_{i_\mu}(x) = [u_1(x), \ldots, u_n(x)] C_{i_\mu}, \qquad \mu = 1, 2, \ldots, q,$$

and so

$$B(x) = [u_1(x), \ldots, u_n(x)]\, \tilde{C}\delta \equiv 0. \quad \blacksquare$$

We should emphasize that in Lemma 3.1 the ordering is only a convenience. Moreover, the integer q might take on any value greater than 1; i.e., for general n it may be possible to construct local support splines with as few as two one-sided splines and with support on only one interval $[x_l, x_{l+1}]$ (cf. Section 5, Remark 8). For the more usual classes of splines one needs $q > n$, however.

The spline (3.1) is a general analog of the classical B-spline of Curry and Schoenberg [1]. To see the connection with the polynomial spline case, we note that if $L = D^n$, then $L_i = D^i$, $L_i^* = (-1)^i D^i$, and $\mathscr{N} = \mathscr{N}^* = \operatorname{span}\{1, x, \ldots, x^{n-1}\}$.

4. A Finite-Dimensional Space of Splines and a Corresponding Basis

Let $x_0 < x_1 < \cdots < x_k < x_{k+1}$ be prescribed. For $\nu = 1, 2,..., k$ let $\Lambda_\nu = \{\lambda_{\nu i}\}_{i=1}^{l_\nu}$ be EHB-sets of $1 \leqslant l_\nu \leqslant n$ linear functionals with support at x_ν. Let $\mathscr{N}$ be the null space of a differential operator L as in Section 1. With $\Delta = \{x_1, ..., x_k\}$ and $\Lambda = \{\Lambda_1, ..., \Lambda_k\}$ we define

$$\mathscr{S}p(\Delta; \Lambda; \mathscr{N}) = \{s : s|_{I_\nu} = s_\nu|_{I_\nu} \quad \text{for some} \quad s_\nu \in \mathscr{N}, \nu = 0, 1,..., k,$$

$$\lambda_{\nu i} s_{\nu-1} = \lambda_{\nu i} s_\nu, \qquad i = 1, 2,..., l_\nu; \nu = 1, 2,..., k\},$$

where as before, $I_\nu = (x_\nu, x_{\nu+1})$, $\nu = 0, 1,..., k$. We have used the symbol $\mathscr{S}p$ here to distinguish this finite-dimensional space of splines on $[x_0, x_{k+1}]$ from the infinite-dimensional space $\mathscr{S}$ considered earlier.

We begin by showing $\mathscr{S}p$ is $(K + n)$-dimensional, where $K = \sum_{\nu=1}^k m_\nu$ and $m_\nu = n - l_\nu$, $\nu = 1, 2,..., k$. In fact, we provide a basis for $\mathscr{S}p$. Let $m_0 = n$, and define

$$C_{0j} = \begin{bmatrix} L_{j-1}^* v_1(x_0) \\ L_{j-1}^* v_n(x_0) \end{bmatrix}, \qquad j = 1, 2,..., n.$$

Now let $\{\rho_{\nu j}(x)\}_{j=1,\nu=0}^{m_\nu,k}$ be defined by (2.9) and (2.10).

Theorem 4.1. *$\mathscr{S}p(\Delta; \Lambda; \mathscr{N})$ is $(K + n)$-dimensional ($K = \sum_{\nu=1}^k m_\nu$) and is spanned by*

$$\{\rho_{\nu j}(x)\}_{j=1,\nu=0}^{m_\nu,k}. \tag{4.1}$$

Proof. Every spline $s \in \mathscr{S}p$ has a representation

$$s(x) = \left\{ s_\nu(x) = \sum_{j=1}^n \beta_{\nu j} u_j(x) \right\}, \qquad x \in I_\nu, \nu = 0, 1,..., k.$$

Since the $n(k + 1)$ coefficients are constrained by $\sum_{\nu=1}^k l_\nu$ linearly independent linear conditions, we conclude that $\mathscr{S}p$ is of dimension $n + K$. By Lemma 2.1, $\{\rho_{\nu j}\}_{j=1}^{m_\nu} \subset \mathscr{S}$ and are linearly independent, $\nu = 1, 2,..., k$. Clearly, $\{\rho_{0j}\}_{j=1}^{m_0} \subset \mathscr{S}$, and are linearly independent since $C_{01}, ..., C_{0n}$ are linearly independent. It remains to show that the entire collection (4.1) is linearly independent.

Suppose $0 \equiv \sum_{\nu=0}^k \sum_{j=1}^{m_\nu} d_{\nu j} \rho_{\nu j}$. Then $0 \equiv \sum_{j=1}^{m_0} d_{0j} \rho_{0j}(x)$ for $x < x_1$ (since the other ρ's are zero by their support properties). By the aforementioned linear independence of $\{\rho_{0j}\}_{j=1}^{m_0}$, we conclude that $d_{01} = \cdots = d_{0n} = 0$. We are left with $0 = \sum_{j=1}^{m_1} d_{1j} \rho_{1j}(x)$ for $x < x_2$, which implies $d_{11} = \cdots = d_{1m_1} = 0$. The process can be continued to show all of the d's are 0. ∎

We write $\rho_1, \ldots, \rho_{K+n}$ for the lexicographical ordering of the splines (4.1), and let $C_1, \ldots, C_{K+n}$ be the corresponding ordering of the $C_{\nu j}$ for which (2.9) holds. Our aim is to construct local support bases for $\mathscr{S}p$ by taking linear combinations of $\rho_1, \ldots, \rho_{K+n}$. First we note

LEMMA 4.2. *Let $\beta_1, \ldots, \beta_r$ be r linearly independent vectors in $\mathbb{R}^{K+n}$. With $\beta_\nu = (\beta_{\nu 1}, \ldots, \beta_{\nu K+n})$, let*

$$B_\nu(x) = \sum_{\mu=1}^{K+n} \beta_{\nu\mu}\rho_\mu(x), \qquad \nu = 1, 2, \ldots, r. \tag{4.2}$$

Then $\{B_\nu\}_{\nu=1}^{r}$ is a set of r linearly independent splines in $\mathscr{S}p$.

Proof. If

$$0 \equiv \sum_{\nu=1}^{r} d_\nu B_\nu = \sum_{\nu=1}^{r} d_\nu \sum_{\mu=1}^{K+n} \beta_{\nu\mu}\rho_\mu = \sum_{\mu=1}^{K+n} \rho_\mu \left(\sum_{\nu=1}^{r} d_\nu \beta_{\nu\mu} \right),$$

then the linear independence of the ρ's (cf. Theorem 4.1) implies $0 = \sum_{\nu=1}^{r} d_\nu \beta_\nu$. But then the linear independence of the β's implies $d_1 = \cdots = d_r = 0$. ∎

Clearly, if we can find $r = K + n$ coefficient vectors in Lemma 4.2, then (4.2) will be a basis for $\mathscr{S}p$. Since we want a local basis, we should choose these coefficient vectors with Lemma 3.1 in mind. The idea is to find $K + n$ linearly independent β's while still keeping the supports of the resulting basis splines small. How successful this will be depends on the properties of the matrix $C = (C_1, \ldots, C_{K+n})$. The following theorem gives a condition on C which suffices to construct a basis for $\mathscr{S}p$ consisting of splines with supports on at most n intervals between the knots. We will use the notation $C\langle i, j\rangle$ to denote the submatrix of C obtained by taking columns i through j, where $1 \leqslant i < j \leqslant K + n$. We also find it convenient to introduce the notation $\epsilon_0 = 0$, $\epsilon_\nu = \epsilon_{\nu-1} + m_{\nu-1}$, $\nu = 1, 2, \ldots, k + 1$.

THEOREM 4.3. *Suppose that, for $\nu = 0, 1, \ldots, k - n$,*

$$C\langle \epsilon_{\nu+1} + 1, \epsilon_{\nu+n+1}\rangle \quad \textit{is of full rank } n. \tag{4.3}$$

Then there exists a basis $\{B_\nu\}_{\nu=1}^{K+n}$ for $\mathscr{S}p$ such that

$$B_{\epsilon_\nu+1}, \ldots, B_{\epsilon_\nu+m_\nu} \textit{ have support on } [x_\nu, x_{\nu+n}], \qquad \nu = 0, 1, \ldots, k - n, \tag{4.4}$$

$$B_{\epsilon_\nu+1}, \ldots, B_{\epsilon_\nu+m_\nu} \textit{ have support on } [x_\nu, x_{k+1}], \qquad \nu = k - n + 1, \ldots, k. \tag{4.5}$$

Proof. For $\nu = 0, 1,..., k - n$ and $j = 1, 2,..., m_\nu$ choose $\beta_{\epsilon_\nu + j}$ to be the $K + n$ vector with $(\epsilon_\nu + j)$th component equal to 1; the $\epsilon_{\nu+1} + 1,..., \epsilon_{\nu+n+1}$ components equal to δ, where δ is any solution of

$$C\langle \epsilon_{\nu+1} + 1, \epsilon_{\nu+n+1}\rangle \delta = -C_{\epsilon_\nu + j};$$

and the remaining components zero. By Lemma 3.1 the corresponding B's (defined by (4.2)) have the stated support properties (4.4). For $\nu = k - n + 1,..., k$, and $j = 1, 2,..., m_\nu$, let $\beta_{\epsilon_\nu + j}$ be a $K + n$ vector with $(\epsilon_\nu + j)$th component equal to 1 and the other components 0. Then

$$B_{\epsilon_\nu + j}(x) = \rho_{\nu, j}(x), \qquad j = 1, 2,..., m_\nu, \ \nu = k - n + 1,..., k,$$

which clearly have the support properties (4.5). Now by construction, $\beta_1,..., \beta_{K+n}$ are patently linearly independent, so by Lemma 4.2, $\{B_\nu\}_{\nu=1}^{K+n}$ is a basis for $\mathscr{S}p$. ∎

A common situation in which the hypothesis (4.3) of Theorem 4.3 holds is the case where $\{v_i\}_1^n$ is a Tchebycheff system and each Λ_ν contains the point evaluation e_{x_ν}, $\nu = 1, 2,..., k$. Indeed, in this case for $\nu = 0, 1,..., n - k$ the matrix $C\langle \epsilon_{\nu+1} + 1, \epsilon_{\nu+n+1}\rangle$ contains the submatrix $V_{\nu+1} = (v_i(x_j))_{i=1, j=\nu+1}^{n, \nu+n}$ which is rank n by the Tchebycheff property.

With stronger hypotheses on C, a basis for $\mathscr{S}p$ can be found with significantly smaller supports. We illustrate this with the following theorem which deals with the case where each of the sets Λ_ν is a Hermite set. We say $\Lambda_\nu = \{\lambda_{\nu j}\}_{j=1}^{l_\nu}$ is a Hermite set of linear functionals with support at x_ν provided $\lambda_{\nu j} = e_{x_\nu}^{(j-1)}$ for $j = 1, 2,..., l_\nu$. In this case $C_{\nu j} = [L_{j-1}^* v_1(x_\nu),..., L_{j-1}^* v_n(x_\nu)]^T$ and $\rho_{\nu j}(x) = \hat{\theta}_{n-j}(x; x_\nu)$, $j = 1, 2,..., m_\nu$, $\nu = 0, 1,..., k$.

THEOREM 4.4. *Suppose $\{v_i\}_{i=1}^n$ is an Extended-Tchebycheff* (ET-) *system on $[x_0, x_{k+1}]$. In addition, suppose for $\nu = 1, 2,..., k$ that Λ_ν is a Hermite set of linear functionals with support at x_ν. Then there exists a basis $\{B_\nu\}_{\nu=1}^{K+n}$ for $\mathscr{S}p$ such that*

$$B_{\epsilon_\nu + j} \text{ has support on } [x_\nu, x_\mu], \qquad j = 1, 2,..., m_\nu, \quad \nu = 0, 1,..., k, \tag{4.6}$$

where $\mu = \mu(\nu, j) = \min\{\omega: \sum_{i=\nu+1}^{\omega} m_i \geqslant n - j + 1\}$.

Remark. Let $y_1,..., y_{K+2n}$ be the nondecreasing ordering of $x_0,..., x_0$, $x_1,..., x_1,..., x_k,..., x_k$, $x_{k+1},..., x_{k+1}$, where for each $\nu = 0, 1,..., k + 1$, the number x_ν is repeated m_ν times ($m_{k+1} = n$). Then the above support properties can be stated as

$$B_i \text{ has support on } [y_i, y_{i+n}], \qquad i = 1, 2,..., K + n. \tag{4.7}$$

Proof. Let $C_{K+n+j} = C_j$, $j = 1, 2,..., n$. Fix $0 \leqslant \nu \leqslant k$ and $1 \leqslant j \leqslant m_\nu$, and let $\delta \in \mathbb{R}^n$ be a solution of

$$(C_{i_1}, C_{i_2},..., C_{i_n})\delta = -C_{\epsilon_\nu+j}, \tag{4.8}$$

where $(i_1, i_2,..., i_n) = (\epsilon_\nu + 1,..., \epsilon_\nu + j - 1, \epsilon_{\nu+1} + 1,..., \epsilon_{\nu+1} + n - j + 1)$. The matrix of the system (4.8) is nonsingular because it is equivalent (in the sense of similar matrix representations produced by elementary column operations) to

$$\begin{bmatrix} v_1(x_\nu) & \cdots & v_1^{(j-2)}(x_\nu) & v_1(x_{\nu+1}) & \cdots & v_1(x_\mu) & \cdots & v_1^{(r_\mu-1)}(x_\mu) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ v_n(x_\nu) & \cdots & v_n^{(j-2)}(x_\nu) & v_n(x_{\nu+1}) & \cdots & v_n(x_\mu) & \cdots & v_n^{(r_\mu-1)}(x_\mu) \end{bmatrix}, \tag{4.9}$$

where $1 \leqslant r_\mu \leqslant m_\mu$ is such that $j - 1 + \sum_{i=\nu+1}^{\mu-1} m_i + r_\mu = n$. The matrix (4.9) is nonsingular in view of the ET-property of the $\{v_i\}_1^n$, while the equivalence of the matrix in (4.8) with (4.9) follows from the fact that

$$C_{\epsilon_\nu+i} = (L_{i-1}^* v_1(x_\nu),..., L_{i-1}^* v_n(x_\nu))^T,$$

and the fact that $L_{i-1}^* v$ involves a linear combination of $v,..., v^{(i-2)}$, (cf. (2.1)). Now the required B-splines are given by

$$B_{\epsilon_\nu+j}(x) = \rho_{\epsilon_\nu+j}(x) + \sum_{\mu=1}^{n} \delta_\mu \rho_{i_\mu}(x).$$

(In the sum we ignore terms with $i_\mu \geqslant K + n$.) ∎

5. Remarks

1. For polynomial splines with Hermite ties at the knots, local support splines (B-splines) were introduced in 1947, though not published until 1966, by Curry and Schoenberg [1]. They are obtained by taking appropriate (confluent) divided differences of the kernel $(x - \xi)_+^{n-1}/(n-1)!$. Using a form of generalized divided difference, Karlin [11] constructed local support splines in the case where $\mathscr{N}$ is spanned by an Extended-Complete-Tchebycheff (ECT-) system (i.e., $\mathscr{N}$ is the null space of a differential operator L with Pólya's property W), and where the Λ_ν are Hermite sets. The first local bases for splines with more general ties were obtained in our earlier paper [10], where we considered g-splines; i.e., the case where $L = D^{2n}$ and the Λ_ν are Hermite–Birkhoff sets consisting of Hermite sets through order $n - 1$ coupled with certain higher-order natural boundary functionals.

2. Greville [6] considered a class of splines which is the special case of $\mathscr{S}$ where $\Lambda_\nu = \{e_{x_\nu}^{(j)}\}_{j=0}^{n-2}$; that is, the splines were forced to be in C^{n-2} globally.

Local support bases were not studied there, however. Classes of splines as general as $\mathscr{S}$ do not seem to have been dealt with in the literature to date. Moreover, much of the development here can be carried over to still more general classes of splines.

3. It should be emphasized that the results here are truly constructive; i.e., the appropriate matrices can be set up and the coefficients of local support splines can be determined computationally. It is clear that Theorems 4.3 and 4.4 lead directly to algorithms. One can also envision algorithms based on Lemma 4.2 which examine numerically the structure of C, and seek to obtain a basis for $\mathscr{S}p$ with small supports.

4. There is no need to review at length the usefulness of local support bases in applications such as the finite element method, etc. Some applications are indicated in the papers [2–4; 8–10; 11, Chap. 10; 13–14].

5. The classical (polynomial) B-splines, (and the analogs developed by Karlin; cf. Remark 6) enjoy a variety of important special properties, including, for example, the facts that they are positive and that they can be computed conveniently by recursions (cf. [2]). Unfortunately, many of these properties are lost for the general case. Thus, for example, the local support splines constructed here are not usually positive. There remain interesting open questions as to how to construct local support bases to preserve such properties. Ferguson [5] has recently investigated the question of when local support g-splines can be positive.

6. When $\Lambda_\nu = \{e_{x_\nu}\}$ in Theorem 4.4, we need only assume that $\{v_i\}_1^n$ is a T-system, rather than that it is an ET-system. We should point out that when $\{u_i\}_1^n$ form an ET-system (and the Λ_ν are Hermite sets as in Theorem 4.4), the construction of Karlin [11] can be used. Although Karlin's construction was carried out for $\{u_i\}_1^n$ an ECT-system, it is known (Karlin and Studden [12, p. 242]) that given any ET-system on an interval $[a, b]$, there is an ECT-system which spans the same space on $[a, b)$. Since Karlin's approach yields local support bases which are positive (even totally positive; see [11]), it should be preferred over the construction of Theorem 4.4. We also note that the construction of Theorem 4.4 is essentially equivalent to that of Schmidt and Lancaster [14], although they obtained their bases only for the case where L has constant coefficients. (In that case $v_i = u_i$, $i = 1, 2, \ldots, n$, and $\hat{\theta}(x; \xi)$ is a translation kernel.)

7. Often, one is interested in spaces of splines with special end conditions (e.g., natural splines, type I splines, etc.) or in periodic splines (where, e.g., in the definition of $\mathscr{S}p$ we identify x_0 and x_{k+1} and require $s_0 = s_k$). Bases for such spaces of splines are easily obtained from the results of Section 4 by enforcing the special end conditions, or for the periodic case by slight modifications of the methods of Section 4.

8. Let $n = 2$, $L = D^2$, $\mathcal{N} = \{1, x\}$, and $\Lambda_\nu = \{e'_{x_\nu}\}_1^k$. Then $\mathcal{S}$ is the class of piecewise linear functions with equal slopes in all the intervals between the knots. (The linear pieces are *not* required to match continuously at the knots.) It is easily seen that $\theta_1(x; \xi) = (x - \xi)_+^1$ and $\hat{\theta}_0(x; \xi) = (x - \xi)_+^0$. With each knot we have associated one basic function $\rho_\nu(x) = (x - x_\nu)_+^0$. Now if we look at the space $\mathcal{S}p$ we see that it is $(k + 2)$-dimensional and is spanned by 1, x, and $\{(x - x_\nu)_+^0\}_{\nu=1}^k$. Lemma 3.1 permits the construction of local support splines; e.g., $B(x) = (x - x_\nu)_+^0 - (x - x_{\nu+1})_+^0$. (Note: these are a linear combination of only two functions.) There is, however, no local support basis for $\mathcal{S}p$. Indeed, if a spline $s(x)$ vanishes on some interval, then it must have zero slope in all the intervals. Thus no collection of local support splines can represent x, which is, of course, in $\mathcal{S}p$. The problem is that

$$C = \begin{bmatrix} -x_0 & 1 & \cdots & 1 \\ x_1 & -0 & \cdots & -0 \end{bmatrix},$$

since $v_1 = -x$, $v_2 = 1$, and $L_1^* = -D$.

APPENDIX

The following result may be known, but we have been unable to find it.

LEMMA. *If* (2.2) *holds, then so does* (2.3).

Proof. We proceed by induction. For $j = n - 1$, (2.3) is (2.2). Now suppose (2.3) holds for $j = n - 1, \ldots, \tilde{j}$, where $n - 1 \geqslant \tilde{j} \geqslant 1$. Differentiating (2.3) for $\tilde{j}$ yields

$$-\sum_{i=1}^{n} u_i^{(k)}(x)\, DL_{n-\tilde{j}-1}^* v_i(x) = \sum_{i=1}^{n} u_i^{(k+1)}(x) L_{n-\tilde{j}-1}^* v_i(x).$$

Adding $\sum_{i=1}^{n} u_i^{(k)}(x)\, v_i(x)\, a_{\tilde{j}}(x) = \delta_{k,n-1} a_{\tilde{j}}(x)$ yields

$$\sum_{i=1}^{n} u_i^{(k)}(x) L_{n-\tilde{j}}^* v_i(x) = \sum_{i=1}^{n} u_i^{(k+1)}(x) L_{n-\tilde{j}-1}^* v_i(x) + a_{\tilde{j}}(x)\delta_{k,n-1}. \qquad \text{(a.1)}$$

Since $\delta_{k,\tilde{j}-1} = \delta_{k+1,\tilde{j}}$, we have proved (2.3) for $j = \tilde{j} - 1$ and $k = 0, 1, \ldots, n - 2$. Now, for $k = n - 1$, we substitute $u_i^{(n)}(x) = -\sum_{\nu=0}^{n-1} a_\nu(x)\, u_i^{(\nu)}(x)$ in (a.1) to obtain

$$\begin{aligned} \sum_{i=1}^{n} u_i^{(n-1)}(x) L_{n-\tilde{j}}^* v_i(x) &= -\sum_{i=1}^{n}\sum_{\nu=0}^{n-1} a_\nu(x) u_i^{(\nu)}(x) L_{n-\tilde{j}-1}^* v_i(x) + a_{\tilde{j}}(x) \\ &= -a_{\tilde{j}}(x) + a_{\tilde{j}}(x) = 0. \end{aligned}$$

This completes the induction step and the proof. ▌

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